STATIC AND DYNAMIC ANALYSIS OF LAMINATE BEAMS USING HIGH ORDER SHEAR DEFORMATION THEORY

I. M. B. DUPRET MIRANDA
IST- *Instituto Superior Técnico, Departamento de Engenharia Mecância, Av. Rovisco Pais, 1096 Lisboa Codex, Portugal*
Iabna_miranda@facc.co.at

The present work consists of the development of three finite element models based in high order shear deformation theory (HSDT) that are applied to the static and dynamic analysis of laminated composite beams. The developed models are based on Loja [1] proposals and they consider the mathematical formularization presented by Correia [2]. Fifteen case studies of different loads and boundary conditions were studied. Matlab was the calculus software tool adopted, in consonance with its great academic relevance.

Keywords: High Order Shear Deformation Theory –HSDT, Mindlin-Timoshenko Theory, Euler-Bernoulli Theory, Non-Linear Shear Deformation, MultiLayered Composite Beams, Finite Element Method, Static Analysis, Dynamic Analysis, Free Vibrations, MATLAB.

1 INTRODUCTION

In this new millennium, environmental concerns have taken a major relevance for Earth population, due to politicians' initiative as well as due to the increase of death numbers by nature catastrophic phenomena, which are directly connected to all negative global warming effects.

This way world population has developed a green conscious that constantly seeks for explanation answers as well as new energetic solutions. Facing this new challenge the industry sector is obliged to reorganize its priorities in order to produce a new green and clean energy. For this reason all around the world efforts are being made in optimizing methodologies, construction processes and structural mechanics.

The aeronautical civil world is no exception, because its main income source depends on the trust of the costumer and on the reliability of the entire aeronautical structures. This way we have been watching a significant evolution in infrastructures implantation, as well as in the assembly processes of the airplane itself.

Statistically the airplane is still the safest way of transportation on Hearth but it is also the most pollutant. Engine aeronautical industry as made a long way in the last years seeking for efficiency, struggling against material thermodynamic limitations, such as the maximum allowed temperature on the turbine blades.

It is then clear that it is necessary to consider the Environment natural balance from different points of view, for example, the reduction of fuel consumption per flight hour.

Nowadays we think that the best fuel consumption reduction is achieved by reducing the height of the entire airplane, through structural optimization. In this engineering field the composite materials are the key element due to its structural properties and applications. With them it is possible to produce a high structural stiffness with a significant height reduction, which applied to an entire airplane, has major effects in reducing the fuel consumption and therefore the airplane pollution.

The use of composite materials in civil aviation by the main manufactures is nowadays a reality. The new Airbus A380 has nearly 30% of composite materials and the new airplane from Boeing B78, also known as Dreamliner, with nearly 50% of composite materials. This last one is also famous by its low fuel consumption.

Nevertheless all this industrial development was only possible due to a wide investigation conducted by several scientists, providing all necessary tools to understand this technology.

Their studies revealed that the mechanical behaviour of ply made reinforced composite is strongly dependent of the reinforcement fibber direction. For this reason a laminate structure has to be designed in order to satisfy all particular requests from a specific application, in order to extract the maximum structural advantages from these materials.

Therefore it is crucial to provide to the stress analyst the proper tools, such as optimized numerical models for these specific materials, so that he can perform static, dynamic, fatigue and buckling analysis during the structural design methodology. With this aim in mind we have developed a study in composite laminated beams, based on the high shear deformation theory, with no need of additional correction factors.

Three models have been developed based on the models presented by Loja [1] and implemented in accordance with the mathematical formulation presented by Correia [2].

MATLAB was the software tool used accordingly with its academic relevance in IST. Thanks to this tool it is possible to take advantage of a wide set of interface user friendly tools, witch allows an optimization in the investigation's time quality.

2 DISPLACEMENTS AND STRAIN FIELDS

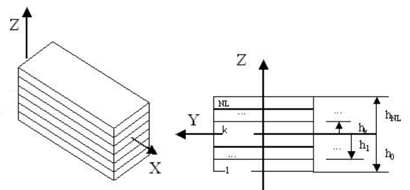

Figure 1-Beam axis

Considering the laminate geometry and the orthogonal referential xz in Fig. 1, by neglecting warping, the axial displacement u and the displacement component w, are expanded by Taylor's series up to the cubic and second power on z-direction respectively. The displacement field can be represented in matrix form as:

u=[Z]q ; u=[u(x,t) w(x,t)]^T ;

$$[Z]=\begin{bmatrix} 1 & 0 & z & 0 & z^2 & 0 & z^3 \\ 0 & 1 & 0 & z^2 & 0 & z & 0 \end{bmatrix};$$

$$q=\{u^0 \quad w^0 \quad \theta_y^0 \quad w^{0*} \quad u^{0*} \quad \beta_z \quad \theta_y^{0*}\}^T; \tag{1}$$

where q is the vector of generalized displacements, representing the appropriated Taylor's series terms defined along the x- axis and z=0. All generalized displacement components are function of time t. The first three terms are related with displacements and rotations as defined on Fig. 1. The remaining parameters are the corresponding higher-order terms representing higher-order transverse cross-sectional deformation modes, which have difficult physical interpretation. Assuming that plane sections remain plane after deformation, but not perpendicular to the geometrical axis one obtains the first-order shear deformation displacement field for the Timoshenko's model (MTT). Further, considering that normals to the reference surface remain normal after deformation. i.e. neglecting transverse shear strains yields $\theta_y^0 = \partial w^0 / \partial x$, then the corresponding Euler-Bernoulli model (EBT) can be formulated, Fig. 2.

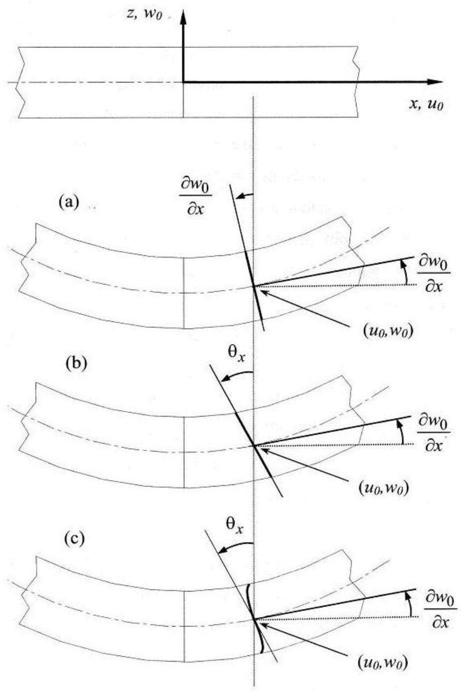

Figure 2-Comparison between deformations due to transverse shear a) Classic theory; b) Mindlin-Timoshenko c) HSDT

Considering the kinematics relations and the HSDT displacement field Eq. (1), the strain field is obtained as.

$$\varepsilon=[\underline{Z}]\varepsilon^0 \; ; \quad \{\varepsilon\}=[\varepsilon_{xx} \quad \varepsilon_{zz} \quad \gamma_{xz}]^T \; ;$$

$$[\underline{Z}]=\begin{bmatrix} 1 & 0 & z & 0 & z^2 & z^3 & 0 & 0 & 0 \\ 0 & 1 & 0 & z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & z & z^2 \end{bmatrix};$$

$$\{\varepsilon_{bm}^*\}^T=\left\{\frac{\partial u}{\partial x} \quad \beta_z \quad \frac{\partial \theta_y^0}{\partial x} \quad 2w^{0*} \quad \frac{\partial u^0}{\partial x} \quad \frac{\partial \theta_y^0}{\partial x} \right.$$

$$\left. \ldots \quad \theta_y^0+\frac{\partial w^0}{\partial x} \quad 2u^{0*}+\frac{\partial(\beta_z)}{\partial x} \quad 3\theta^{0*}+\frac{\partial(w^{0*})}{\partial x}\right\} \tag{2}$$

For the HSDT model, normal stress σ_x and σ_z are considered and a non-linear variation through the thickness is assumed. This nonlinearity is also extended to the shear stress τ_{xz}. Considering the orthogonal referential xz, the constitutive relation for an orthotropic layer, which have an arbitrary fibber orientation, are related to the strains trough the relation:

$$\begin{Bmatrix}\sigma_x\\ \sigma_z\\ \tau_{xz}\end{Bmatrix}_k = \begin{bmatrix}\overline{Q_{11}} & \overline{Q_{13}} & 0\\ \overline{Q_{31}} & \overline{Q_{33}} & 0\\ 0 & 0 & \overline{Q_{55}}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_z\\ \Gamma_{xz}\end{Bmatrix} \tag{3}$$

where the terms of matrix $\overline{Q}$, for the kth layer are given in Reddy [3].

3 FINITE-ELEMENT MODEL

A four node beam element was developed, with seven degrees of freedom per node for static and free vibrations. The displacement field can be represented as.

$$u_e = ZNq_e\ ;$$
$$q_e^i = \{u^0 \quad w^0 \quad \theta_y^0 \quad w^{0*} \quad u^{0*} \quad \beta_z \quad \theta_y^{0*}\}^T\ ;$$
$$\varepsilon_e = \underline{ZB}q_e \tag{4}$$

where N is a matrix with cubic Lagrange functions and q_e the element nodal displacement vector. Hence the strain field is given by.

$$\begin{Bmatrix}\varepsilon_{bm}^*\\ \varepsilon_s^*\end{Bmatrix} = \begin{bmatrix}B_{bm}\\ B_s\end{bmatrix} q^e \tag{5}$$

where B_{bm} and B_s are the transformation matrix between strains and displacements, for bending and shear respectively. The same are given by.

$$B_{bm}^i = \begin{bmatrix}\frac{\partial N_i}{\partial \xi}\frac{1}{J} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & N_i & 0\\ 0 & 0 & \frac{\partial N_i}{\partial \xi}\frac{1}{J} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 2N_i & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{\partial N_i}{\partial \xi}\frac{1}{J} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial N_i}{\partial \xi}\frac{1}{J}\end{bmatrix};$$

$$B_s^i = \begin{bmatrix}0 & \frac{\partial N_i}{\partial \xi}\frac{1}{J} & N_i & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 2N_i & \frac{\partial N_i}{\partial \xi}\frac{1}{J} & 0\\ 0 & 0 & 0 & \frac{\partial N_i}{\partial \xi}\frac{1}{J} & 0 & 0 & 3N_i\end{bmatrix};$$

i=1,…,4

(6)

where J is the Jacobian from the transformation.
By applying Hamilton's variational principle to the total Lagrangean for the eth element, one obtains the following equilibrium equation.

$$K_e q_e + M_e \ddot{q}_e = Q_e \tag{7}$$

where Q_e is the element load vector and $\ddot{q}_e$ is the element acceleration vector. For harmonic vibrations one can obtain:

$$K_e q_e = \omega^2 M_e q_e \tag{8}$$

Where ω is the natural frequency. And for a static solution:

$$K_e q_e = Q_e \tag{9}$$

The element stiffness matrix is.

$$K^e = \int_{-1}^{+1} \left[\underline{B}^T\right]\left(\sum_{k=1}^{NL}\begin{bmatrix}\overline{D}_k^{bm} & 0\\ 0 & \overline{D}_k^{s}\end{bmatrix}\right)\left[\underline{B}\right] Jbd\xi \tag{10}$$

where NL represents the number of plies, b is the width of the beam and J is the Jacobian of the transformation. Matrix $\overline{D}_k^{bm}$ and $\overline{D}_k^{s}$ are given by.

$$\overline{D}_k^{bm} = \int_{h_{k-1}}^{h_k} \{Z_{bm}^T\}\left[\overline{Q}_k^{bm}\right]\{Z_{bm}\}dz\ ;$$

$$\overline{D}_k^{s} = \int_{h_{k-1}}^{h_k} \{Z_s^T\}\left[\overline{Q}_k^{s}\right]\{Z_s\}dz \tag{11}$$

In order to avoid Locking effects the integration referring to shear was made numerically.
The Mass matrix can be obtained by.

$$M^e = \int_{-1}^{+1} [N]^T \left(\sum_{k=1}^{NL} \rho_k \int_{h_{k-1}}^{h} Z_m^T Z_m dz\right)[N] Jbd\xi \tag{12}$$

where h_k is the distance between the medium reference surface and upper surface from ply k, h_{k-1} is the distance between the medium reference surface and lower surface from ply k.

4 APPLICATIONS

Some illustrative numerical results are obtained using the present high order shear deformation theory model HSDT, for static and free vibrations, to show the adequacy to different situations.

4.1 *Static analysis*

4.1.1 Isotropic simply supported rectangular cross-section beam

To study the performance of the HSDT model for different length/thickness ratios a simply supported straight beam type structure subjected to a uniformly distributed loading. p_z=1 N/m was considered. The material and geometrical properties are E=200Gpa (Young Modulus), b=0.01 m (width) and h=0.01 m (thickness). Table 1 shows the maximum transverse displacements in x=L/2 and z=0, obtained with EBT, MTT and HSDT models. A discretization in twenty beam elements was use. It can be observed that the HSDT model predicts the displacements with a good accuracy, even for the lower L/h ratio. For lengths to thickness ratios above 10, all discrete models can predict the maximum transverse displacement with a good precision.

Table 1-Maximum deflection w (m) at z = 0.

Ratio L/h	Elasticity P. stress	EBT Model	$MTT_{(k=5/6)}$ Model	HSDT Model
1	$2.563x10^{-12}$	$7.796x10^{-13}$	$2.655x10^{-12}$	$2.774x10^{-12}$
2	$1.963x10^{-11}$	$1.247x10^{-11}$	$1.999x10^{-11}$	$2.044x10^{-11}$
4	$2.285x10^{-10}$	$1.996x10^{-10}$	$2.298x10^{-10}$	$2.317x10^{-10}$
10	$7.990x10^{-09}$	$7.796x10^{-09}$	$7.995x10^{-09}$	$8.009x10^{-09}$
20	$1.257x10^{-07}$	$1.247x10^{-07}$	$1.256x10^{-07}$	$1.257x10^{-07}$
50	$4.887x10^{-06}$	$4.873x10^{-06}$	$4.882x10^{-06}$	$4.886x10^{-06}$
100	$7.814x10^{-05}$	$7.796x10^{-05}$	$7.800x10^{-05}$	$7.812x10^{-05}$

4.1.2 Clamped laminated beam-box under concentrate load

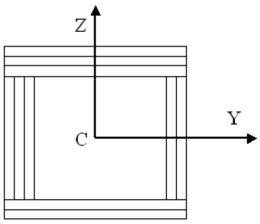

Figure 3 –Beam-box section cut geometry

In this example a beam box with no sectional symmetry was used , Fig. 3. It is a one side clamped beam with a concentrate vertical load, P, on the free side. Its material and geometrical properties are:
$b_{almaesquerda}=b_{almadireita}$=50mm;
$b_{banzosuperior}=b_{banzoinferior}$=80mm;
$h_{almaesquerda}=h_{almadireita}$=20mm; $h_{banzosuperior}$=20mm; $h_{banzoinferior}$=30mm; E=200GPa, P=100N

Table 2- Displacements from beam-box

Ratio L/h	Loja [1] Classic T.	EBT Model	$MTT_{(k=5/6)}$ Model	HSDT Model
2	$2.157x10^{-7}$	$2.156x10^{-7}$	$2.656x10^{-7}$	$2.730x10^{-7}$
4	$3.371x10^{-6}$	$3.369x10^{-6}$	$3.494x10^{-6}$	$3.505x10^{-6}$
10	$2.696x10^{-5}$	$2.695x10^{-5}$	$2.720x10^{-5}$	$2.719x10^{-5}$
20	$2.157x10^{-4}$	$2.156x10^{-4}$	$2.161x10^{-4}$	$2.159x10^{-4}$
50	$3.371x10^{-3}$	$3.369x10^{-3}$	$3.369x10^{-3}$	$3.367x10^{-3}$
100	$2:696x10^{-2}$	$2:695x10^{-2}$	$2:694x10^{-2}$	$2:695x10^{-2}$

The main results obtain are presented in Table 2. A perfect correspondence between EBT Model and the Classic theory present by Loja is observed as expected, because we are comparing outputs obtained with the equivalent theory.
Model MTT and HSDT are presenting higher value solutions because shear is also considered in these models.

4.1.3 Laminated beam under three point bending

The results of the HSDT discrete model were compared with a standard benchmark test of a laminated strip under three point bending, Fig. 4. This test has been has been designed to validate laminate beams. The geometric, loading and mechanical properties are L=0.05m (length), b=10^{-2} m, h=10^{-3} m, P=100N (x=0.025 m), E_1=100GPa, E_3=5GPa, G_{13}=3GPa, ν_{13}=0.4. A discretization of ten elements was used for the test. Table 3 shows the results at middle span, for maximum transverse displacement w, and normal stress σ_x , both evaluated at point E, and shear stress τ_{xz} at point D. A good agreement is obtained with the target results.

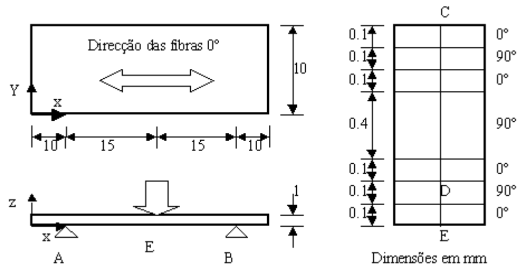

Figure 4 -Laminate beam under three-point bending

Table 3- Maximum deflection and stresses

Model	W(mm)	(MPa)	(MPa)
BENCHmark	-1.060	683.9	-4.1
Loja	-1.059	683.2	-4.6
HSDT	-1.058	683.1	-4.7

4.1.4 Simply supported laminate T-beam

This example presented by Silva et al [14], considers a simply supported laminated composite

T-beam, which was loaded accordingly with Figure 5. The geometric and material properties are:

(ug) – VEER45, R365

E_1=39.25Gpa; E_3=4.5Gpa; h_{layer}=0.35mm; G_{13}=3.0Gpa; ν_{13}=0.29;

(fg) – 1581-ES-67:

$E_1=E_3$=22.5Gpa; h_{camada}=0.24mm; G_{13}=2.85Gpa; ν_{13}=0.28;

Stacking sequence:

Web: $[45°fg/0°ug/45°fg/(0°ug)_2/45°fg/0°ug]_s$

Flange:

$[((0°fg/45°fg)_3/0°fg)_2/0°ug/45°fg/(0°ug)_2/45°fg/0°ug/45°fg]$

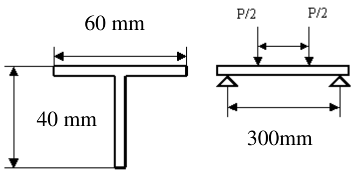

Figure 5 -Laminate T-beam in four point Bending

Silva et al [14], carried out experimental and numerical studies using a commercial finite element program. A discretization in ten HSDT beam elements has been used. The main results are presented in Table 4. One can observe good correspondence between the obtained results.

Table 4- Displacements and strains of T-beam

Model	**Neutral axis**	$\varepsilon_{max} x10^3$	$\varepsilon_{min} x10^3$
EBT	29.83	11.94	-4.07
FEM	28.64	11.00	-4.38
Experimental	30.51	13.08	-4.06
Loja (1995)	30.45	12.27	-3-77
MTT	30.14	10.98	-4.11
HSDT	30.72	12.20	-3.90

4.2 *Free vibrations analysis*

4.2.1 BENCHmark -Isotropic beam clamped-supported

This test was present in BENCHmark. In this example it is used one beam clamped-supported with the following properties: L=10.0m, b=2.0m, h=2.0m E=200GPa, ν_{13}=0.3, ρ=8000Kg/m^2.

For the clamped side the boundary conditions used are: $u^0=w^0=u^{0*}=\theta^0=u^{0*}=\beta_z=\theta_y^{0*}=0$ and for the supported side: $w^0=w^{0*}=\beta_z=0$.

The frequencies for vibration modes are presented in Table 5, for a discretization of twenty elements and a good agreement with the expected values is observed.

Table 5 -Natural Frequencies (Hz)

BENCH mark	**Loja [1]**	**Model MTT**	**Model HSDT**
042.65	040.50	039.58	040.14
148.31	142.37	143.62	148.76
284.55	275.67	250.00	262.37

4.2.2 Orthotropic beam simply supported

In this example a simply supported beam with rectangular section, as presented by Chandrashekhara, was considered. The following material and geometrical properties were used: L=1 in, h=1 in, $E_1=E_3=21.0x10^6$ psi, $G_{13}=0.6x10^6$ psi, ν_{13}=0.3, $\rho=0.13x10^{-3}$eb.s^2.in^{-4}.

In Table 6 is presented a comparison between the natural frequencies obtain by the classic theory, by Chandrashekhara and by Loja [1].

Table 6 – Natural frequencies (KHz), L/h=15

CLT	**Chand. [18]**	**Loja [1]**	**Model MTT**	**Model HSDT**
0.813	0.755	0.755	0.756	0.755
3.250	2.548	2.555	2.495	2.496
7.314	4.716	4.785	4.536	4.557
13.002	6.960	7.201	6.618	6.687
20.316	9.194	9.936	8.681	8.825

Results presented by Chandrashekhara et al [18] were obtained with a first shear deformation theory, while results presented for classic theory can be found in Vinson et al [17]. Loja [1] has also used a HSDT.

Accordingly with Table 6, one can observe a good correspondence between the Models HSDT and MTT and the solution presented by Loja and Chandrashekhara.

4.2.3 Clamped-clamped laminated beam

This example refers to an orthotropic beam, presented by Dipak et al [10]. He used a nine node plate element, based in a high order shear theory. The beam used has the following properties: L=0.1905m, E_1=129.207GPa, E_3=9.425GPa, G_{13}=4.30GPa, ν_{13}=0.3, ρ=1550.06 Kg.m^{-3}.

The multiplier used to adimensionalize the obtain frequencies is:

$$\overline{w} = wL^2\sqrt{\frac{\rho A}{E_3 I}} \tag{11}$$

Table 7- Non-dimensional fundamental frequency

L/h=60	0/90/0/90	0/30/-30/0	0/45/-45/0	0/60/-60/0
EBT	49.58	61.88	55.35	51.03
MTT	62.54	87.86	83.11	78.01
HSDT	54.64	87.42	79.96	69.58
Dipak	55.86	77.95	77.05	76.62
Loja	57.87	72.34	65.25	60.73

As we can see in Table 7, the obtain results reveal good agreements with the results obtain by Dipak and Loja, except for the model EBT.

5 CONCLUSIONS

Three FEM models have been developed for predicting the static and dynamic behaviour of composite laminate beams, representing three different theories: High shear deformation theory, Mindlin–Timoshenko theory and Euler-Bernoulli theory.

With these three models we have studied several beams under different load conditions and DOF constraints.

For the performed analysis one can conclude that the model based on the high order shear deformation theory is the one presenting better results in accordance with the scientific articles published in this area for thick composite beams.

For non thick beams, the models based on Mindlin-Timoshenko and Euler–Bernoulli have also presented consistent results with scientific published data, and since they consume less computation time one can conclude that for these case it is not necessary to use more complex theories.

For dynamic behaviour prediction the high shear deformation theory has presented some advantages in relation to the other two theories, especially for the higher vibration modes.

During this work several application tools have been development in the MATLAB software platform used. Its use has revealed the major aspects from this software and has significantly increased the investigation's time quality.

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